Combinatorics in string topology

Joint work with T. Tradler

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Topology Seminar
January 20, 2023

Main theorem

Theorem (P.–Tradler, in progress)

A slide complex of short-branched trees is a decomposition of the Stasheff quotient of an assocoipahedron of directed planar trees.

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loop space "intersection" trees

Theorem (P.–Tradler)

A slide complex of spaces of short-branched trees is a decomposition of the Stasheff quotient of an assocoipahedron of directed planar trees.

Hochschild "intersection" trees

Plan

Theorem (P.–Tradler, in progress)

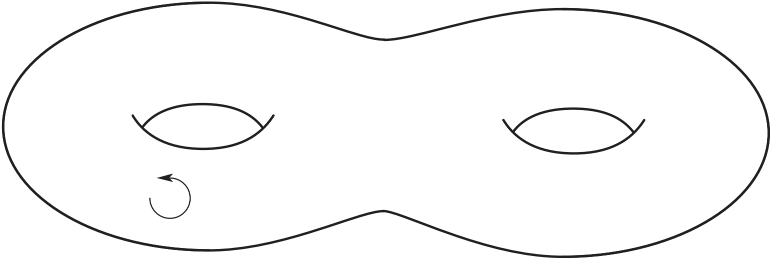
A slide complex of short-branched trees is a decomposition of the Stasheff quotient of an assocoipahedron of directed planar trees.

- Chapter 1: Background
- Chapter 2: Loop space side
- Chapter 3: Hochschild side
- Chapter 4: Main theorem
- Epilogue: Moduli space conjecture

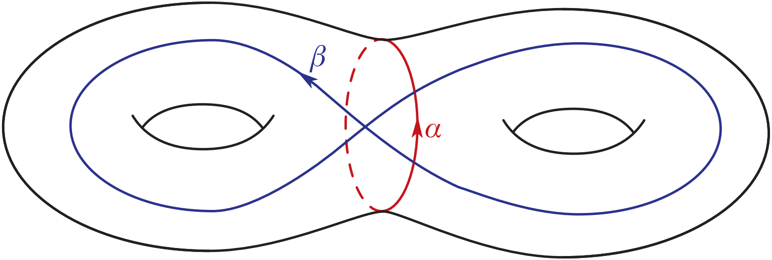
Chapter 1

Background

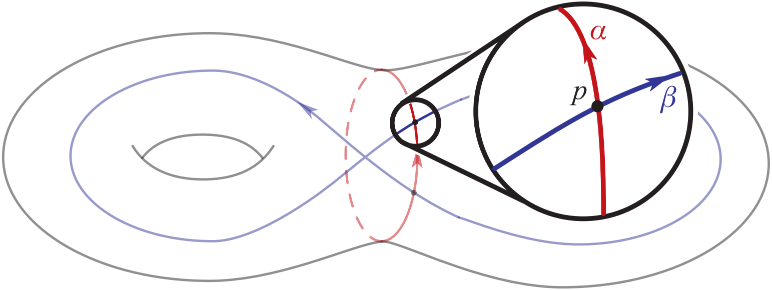
Fix an oriented surface Σ.



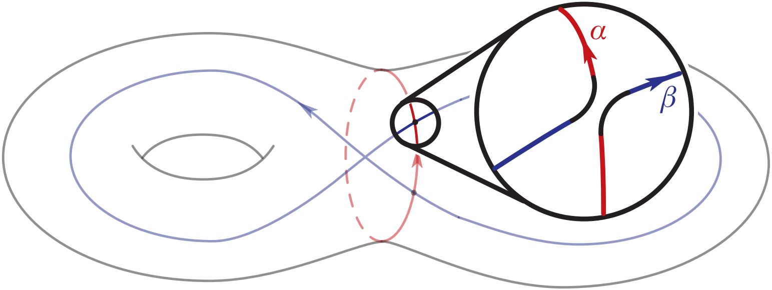
Consider two free homotopy classes α and β of closed curves on Σ.



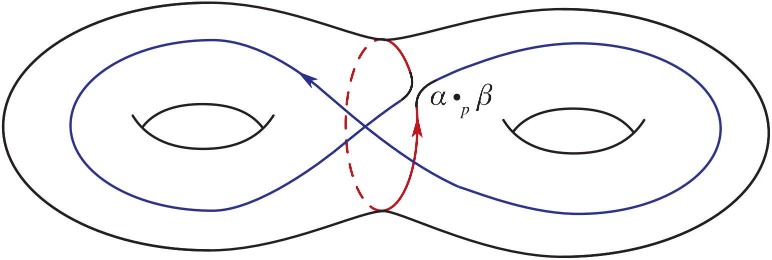
Consider representative curves that intersect one another only in transverse double points p.



Cut α and β at *p* and reconnect the strands in the other way that respects their orientation.



Let $\alpha \cdot_p \beta$ be the closed curve obtained by cutting and reconnecting.



The Goldman Bracket

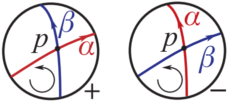
Each intersection point *p* of α and β gives a free homotopy class of closed curves $\alpha \cdot_p \beta$.

Let *H* be the $\mathbb{Q}$-vector space generated by the set of free homotopy classes of closed curves on Σ. (In general, *H* is infinite dimensional.)

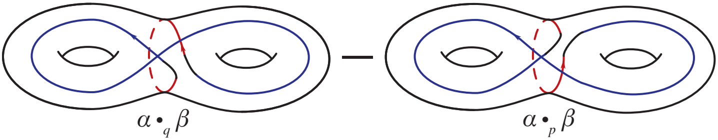
Define

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \pm \alpha \cdot_p \beta.$$

Signs depend on the orientation of Σ



$[\alpha, \beta] =$



The Goldman Bracket

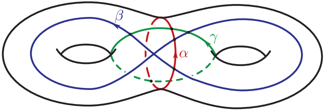
Definition (Goldman Bracket)

Extend $[\ ,\ ]$ linearly to obtain a map $[\ ,\ ] : H \otimes H \longrightarrow H$.

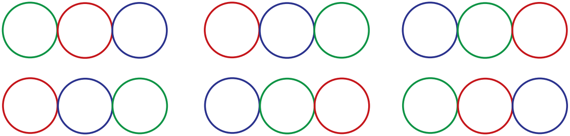
Theorem (Goldman)

The bracket is well defined and gives H the structure of a Lie algebra.

Idea of proof of Jacobi identity: terms cancel in pairs.

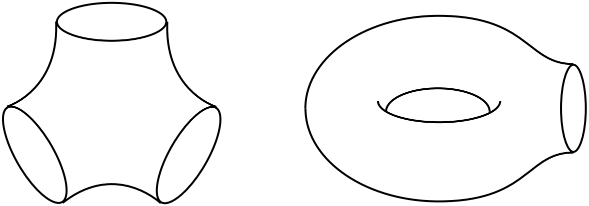


$[[\alpha, \beta], \gamma]$ $[[\beta, \gamma], \alpha]$ $[[\gamma, \alpha], \beta]$



The Goldman Bracket

The Goldman bracket $[\ ,\ ] : H \otimes H \to H$ extends to surfaces with boundary.



Theorem (Gadgil 2011)

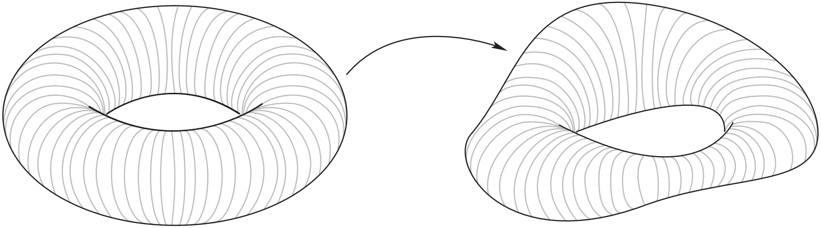
Let $f : \Sigma \to \Sigma'$ be a homotopy equivalence of surfaces with boundary. Then f is homotopic to a homeomorphism if and only if it respects the Goldman bracket.

Let M be a closed, oriented d-dimensional manifold.

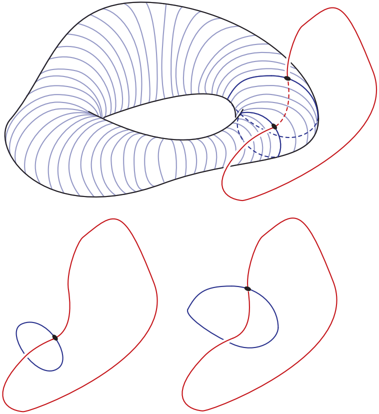
Let $d = 3$.

Let

- H_0 be the $\mathbb{Q}$-vector space generated by free homotopy classes of loops in M.
- H_1 be the $\mathbb{Q}$-vector space generated by homotopy classes of fibered tori in M.



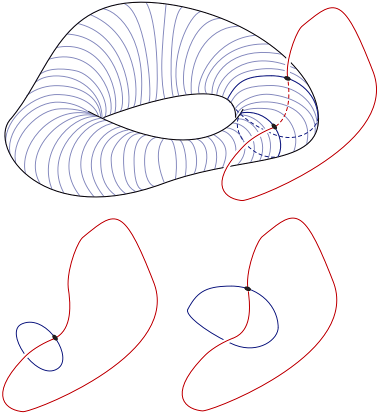
Intersections

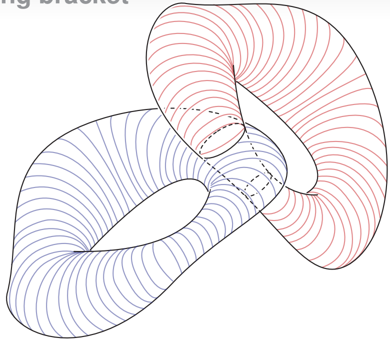


$$H_0 \otimes H_1 \longrightarrow H_0$$

String bracket

Intersections





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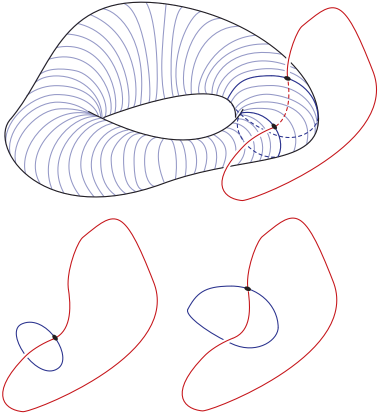

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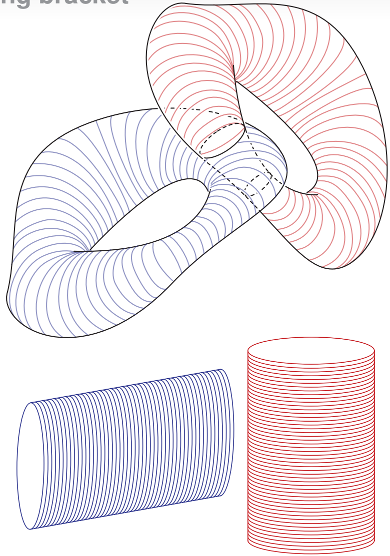

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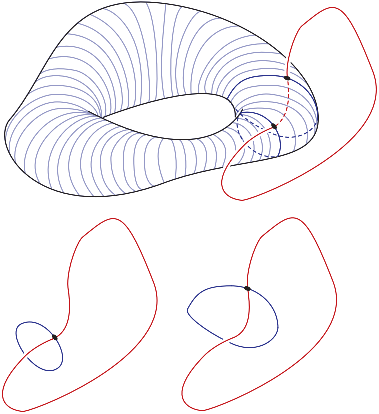

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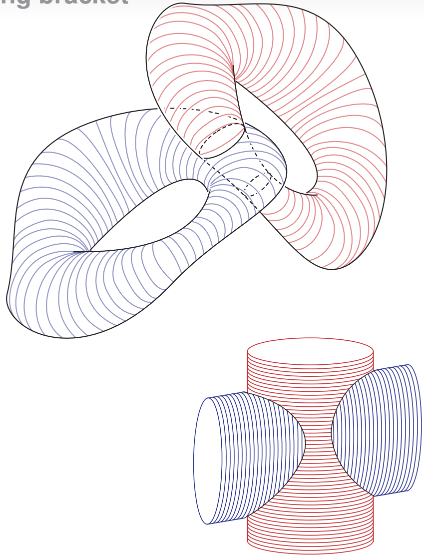

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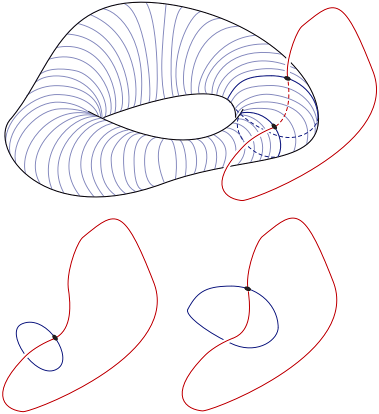

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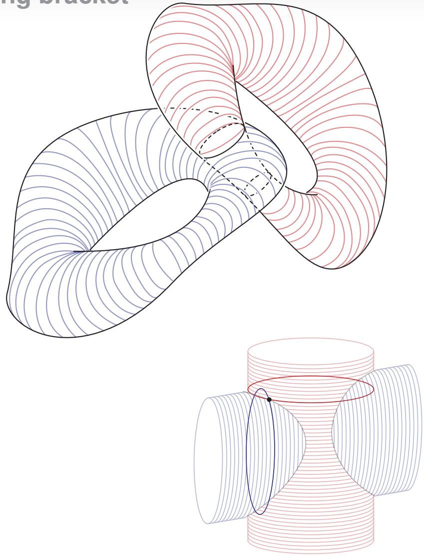

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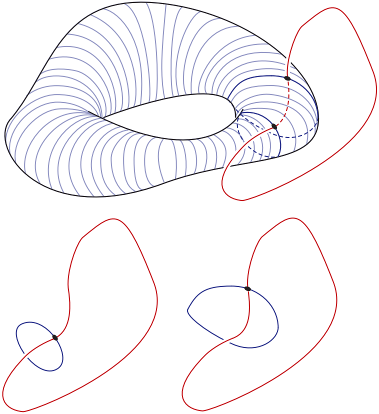

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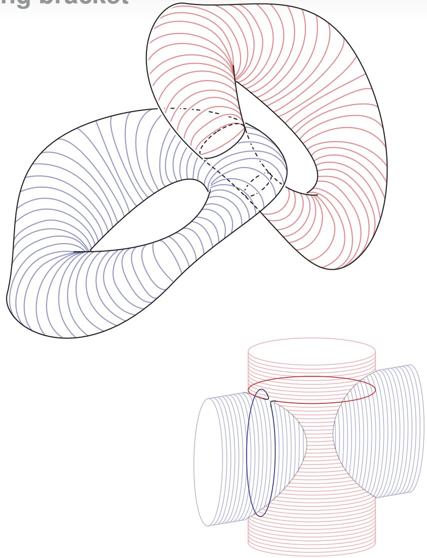

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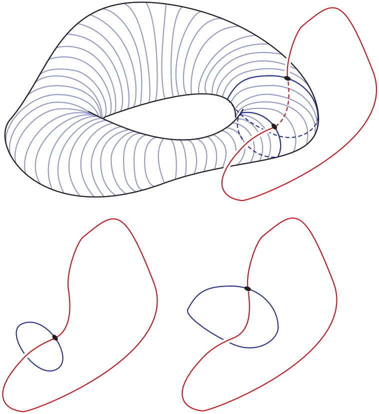

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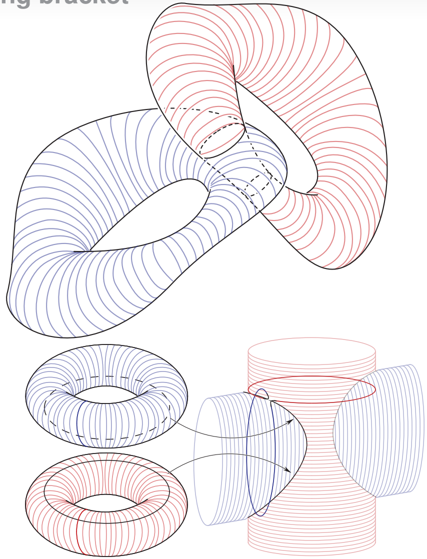

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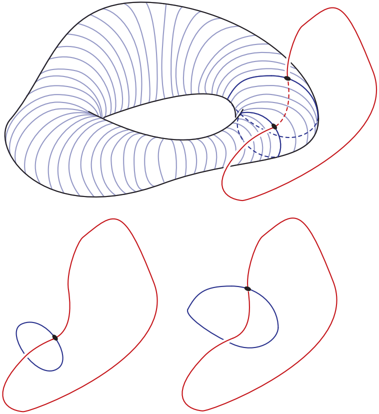

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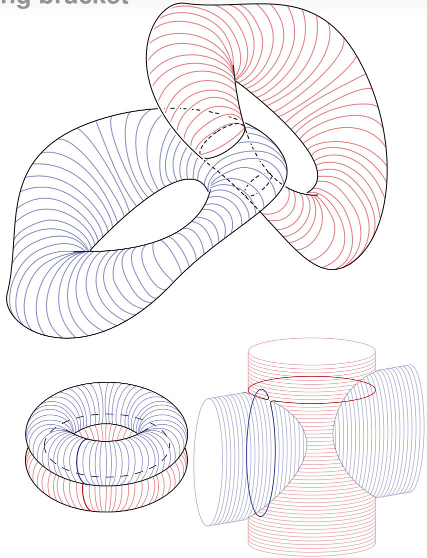

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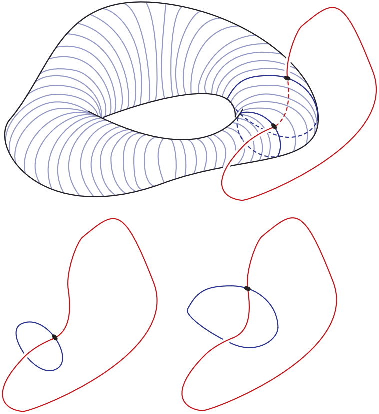

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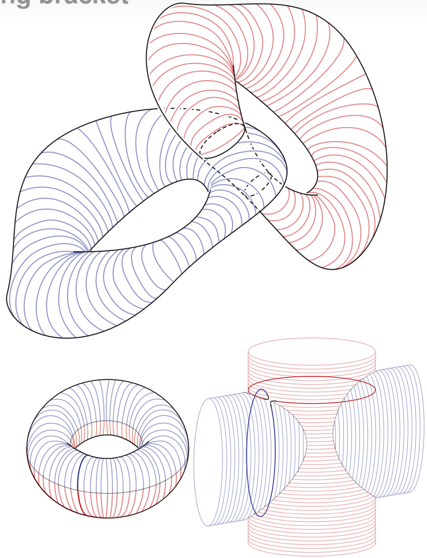

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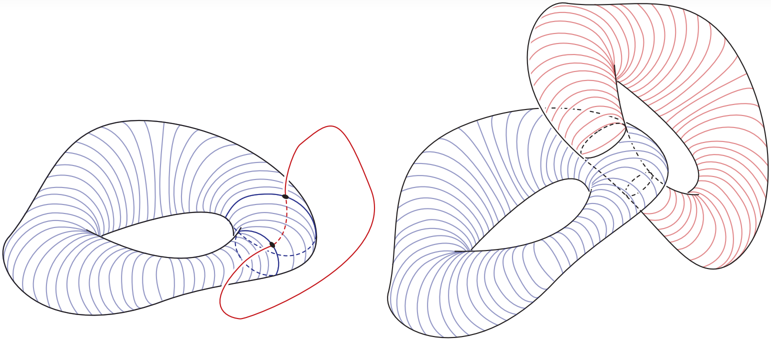


The string bracket for d-dimensional manifolds M is defined analogously.

Theorem (Chas-Sullivan)

Let M be a closed, oriented d-dimensional manifold, let $LM = Maps(S^1, M)$ be its free loop space and let $H_*^{S^1}(LM)$ be the S^1-equivariant homology of LM. Then the string bracket gives $H_*^{S^1}(LM)$ the structure of a graded Lie algebra. When $d = 2$ and $* = 0$, then the string bracket coincides with the Goldman bracket.

In the beginning...

Theorem (Chas–Sullivan 1999)

Let M be an oriented manifold and let $LM = \text{Maps}(S^1, M)$ be its free loop space. The loop product on singular homology $H_*(LM)$ is commutative.

(The loop product is used to define the string bracket.)

Theorem (Gerstenhaber 1963)

Let A be an associative algebra. The cup product on Hochschild cohomology $HH^*(A)$ is commutative.

Theorem (Cohen–Jones 2002)

Let $C^*(M)$ be the cochain algebra of a simply connected, oriented manifold M. There is an isomorphism $H_*(LM) \xrightarrow{\sim} HH^*(C^*(M))$ respecting these products.

Since then...

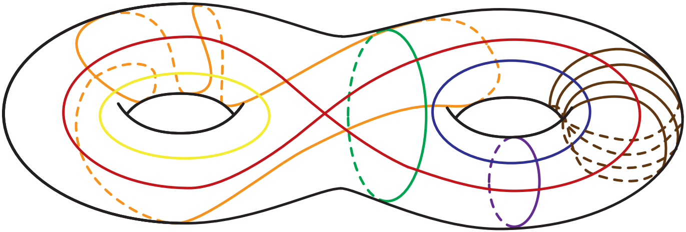
Well actually...

(Abbaspour, Chas, Chataur, Cohen, Costello, Felix, Gerstenhaber, Godin, Kaufmann, McClure, Merkulov, Rivera, Smith, Sullivan, Tamarkin, Tomas, Tradler, Vallette, Voronov, Wahl, Westerland, Zeinalian, etc, etc...)

- There are much richer algebraic structures on the loop space side and on the Hochschild side (paths, loops, and strings; cyclic and non-cyclic versions)

$$H^{\otimes k} \to H^{\otimes \ell}$$

String topology operations



Cutting and reconnecting at intersection points yields generalized operations

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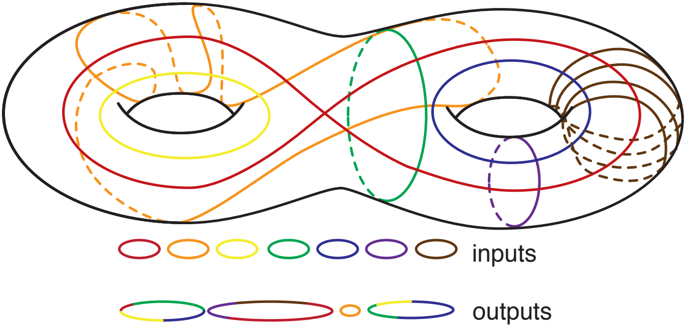

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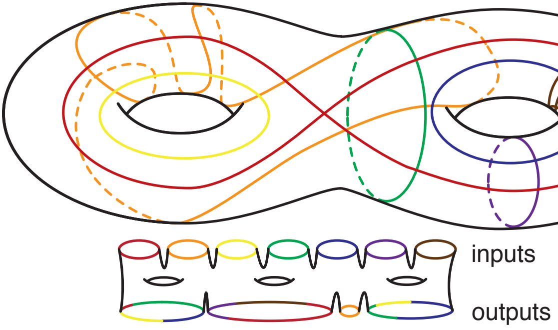

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- There are much richer algebraic structures on the loop space side and on the Hochschild side (paths and strings; cyclic and non-cyclic versions)
$$H^{\otimes k} \to H^{\otimes \ell}$$
- It is better to look for invariants at the chain level than at the homology level
$$C^{\otimes k} \to C^{\otimes \ell}$$

A (vague) motivating question

What is the "best" algebraic structure that is "preserved" under such an isomorphism
$$H_*(LM) \xrightarrow{\sim} HH^*(C^*(M))?$$

Relevant examples

Theorem (Drummond-Cole–P.–Rounds)

Let M be a closed, oriented manifold and let LM be its free loop space.

- *The space of string diagrams $\mathcal{SD}$ parametrizes chain-level string topology operations*

$$C_*(\mathcal{SD}) \longrightarrow \prod_{k,\ell} \mathsf{Hom}(C_*(LM)^{\otimes k}; C_*(LM)^{\otimes \ell}).$$

- *The homology $H_*(\mathcal{SD})$ has the structure of a properad and $H_*(LM)$ is an algebra over this properad (in progress).*

Theorem (Tradler–Zeinalian)

Let A be a $\mathcal{V}_\infty^{(d)}$-algebra and let $CH^*(A)$ be its Hochschild cochain complex.

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Definition debt

Theorem (P.–Tradler, in progress)

A slide complex of short-branched trees is a decomposition of the Stasheff quotient of an assocoipahedron.

Loop space definitions

(1) short-branched trees
(2) string diagrams $\mathcal{SD}$
(3) space of short-branched trees
(4) slide complex

Hochschild definitions

(0′) $\mathcal{V}_\infty^{(d)}$-algebra
(1′) directed planar trees
(2′) directed graphs $\mathcal{DG}_*$
(3′) assocoipahedron
(4′) Stasheff quotient

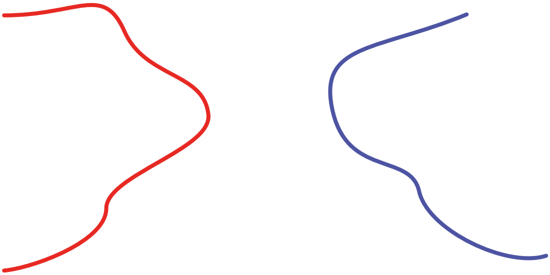
Chapter 2

Loop space side

Chain-level operations

Assume that M is an oriented Riemannian manifold with injectivity radius ε. A Pontriyagin-Thom construction combined with “geodesic concatenation” produces a 2-to-1 operation

$$C_*(LM) \otimes C_*(LM) \to C_*(LM).$$



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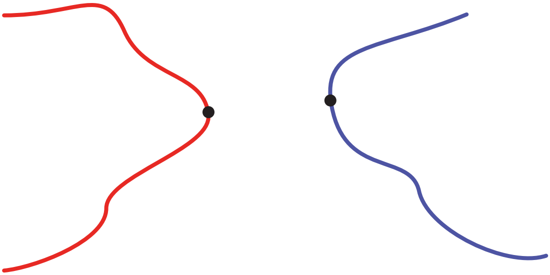

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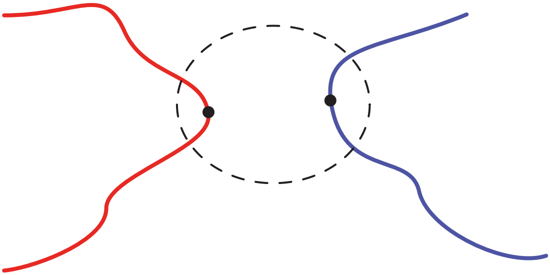

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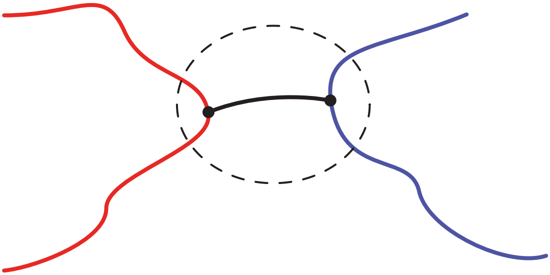

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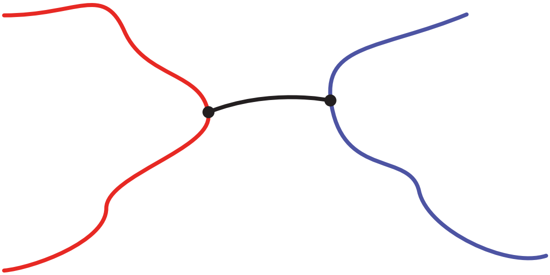

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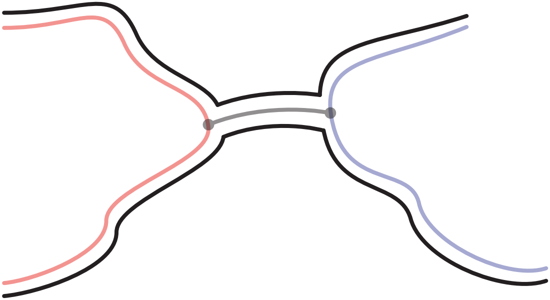
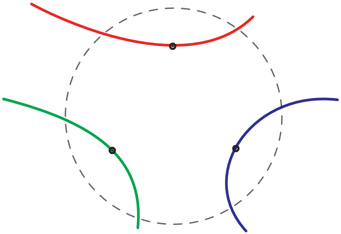


Generalizing geodesic concatenation

Would like a combinatorial object to help generalize this operation.



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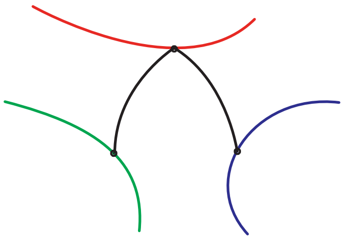

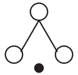

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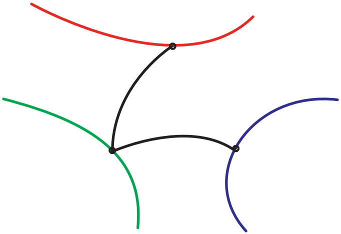

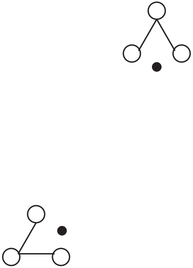

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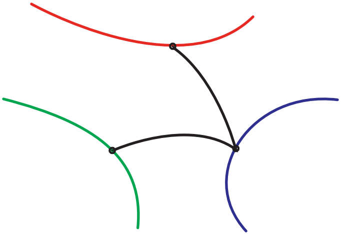
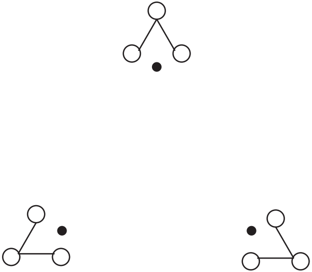
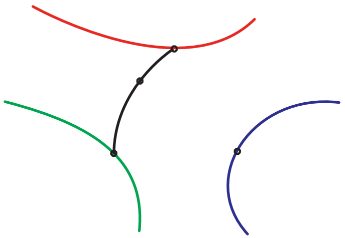
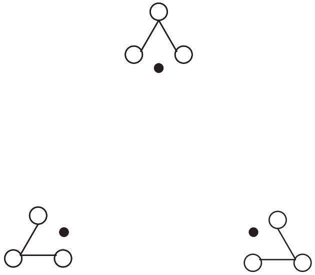




Generalizing geodesic concatenation

Would like a combinatorial object to help generalize this operation.





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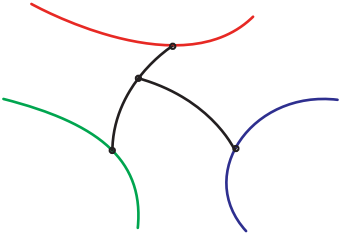

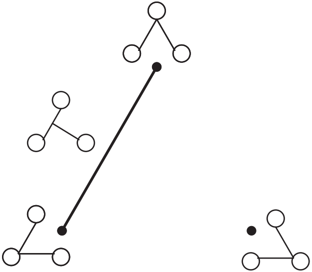

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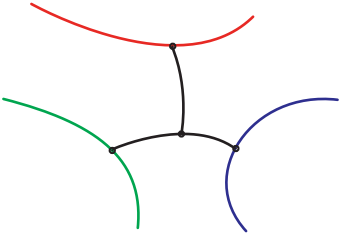

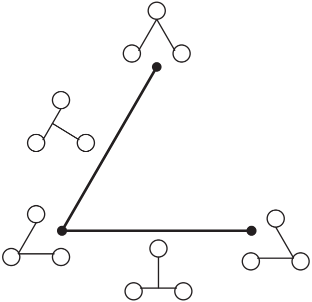

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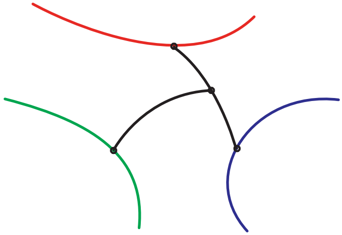

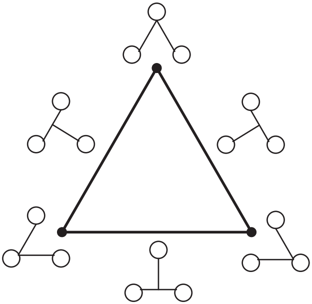

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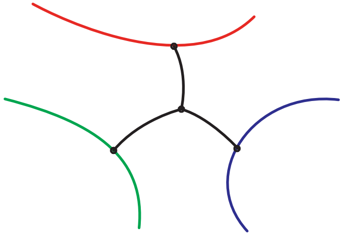
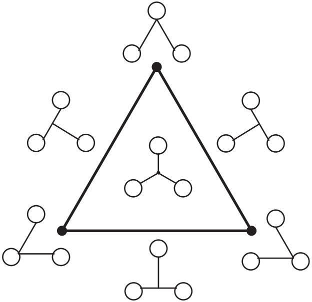


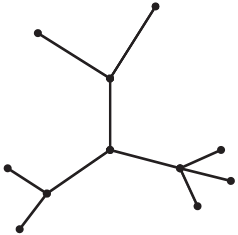


Short-branched trees

Definition 1

A *short-branched tree* is a metric fatgraph tree such that

- its total length is one less than the number of leaves, and
- the total length of each branch is at most one less than its number of leaves.



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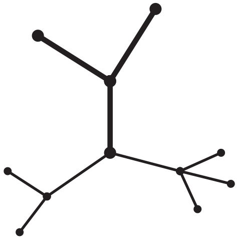

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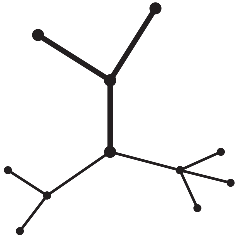
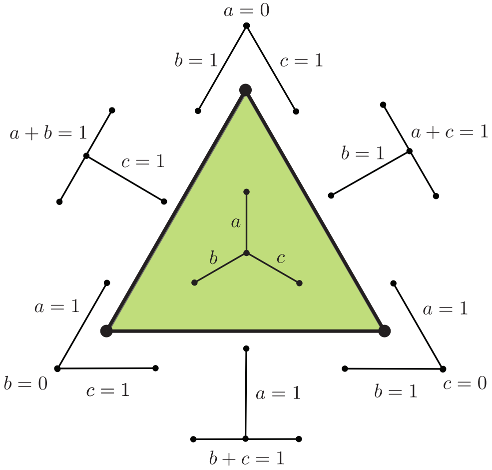


Proposition + Definition 3 (Drummond-Cole–P.–Rounds)

The space of short-branched structures on a fatgraph tree T is a convex polyhedron. Faces correspond to

- *lengths of edges shrinking to zero, or*
- *branches growing to their maximum total length ("breaking").*

Example

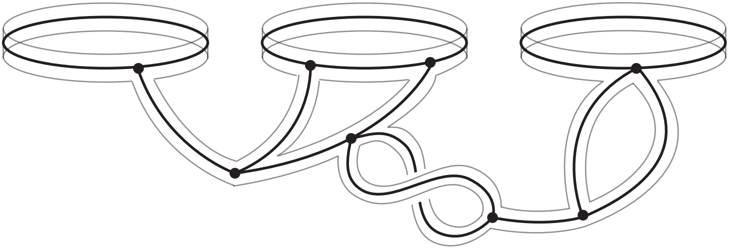


String diagrams

Short-branched trees are building blocks of string diagrams. They keep track of "generalized concatenation configurations" in the loop space.

Definition 2

A *string diagram* is a metric fatgraph constructed from disjoint "input" circles with a collection of short-branched trees attached inductively along their leaves.



The space of string diagrams $\mathcal{SD}$ is the space of chain-level loop-space operations

$$C_*(LM)^{\otimes k} \to C_*(LM)^{\otimes \ell}$$

Definition debt

Loop space definitions

✓ (1) short-branched trees
✓ (2) string diagrams $\mathcal{SD}$
✓ (3) space of short-branched trees
(4) slide complex

Hochschild definitions

(0′) $\mathcal{V}_{\infty}^{(d)}$-algebra
(1′) directed planar trees
(2′) directed graphs $\mathcal{DG}_*$
(3′) assocoipahedron
(4′) Stasheff quotient

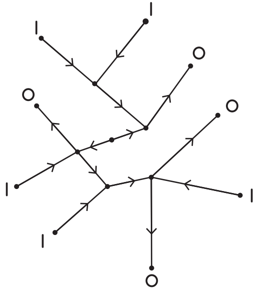
Chapter 3

Hochschild side

Directed planar trees

Definition 1′

A *directed planar tree* is a directed planar tree such that
- every interior vertex has at least one outgoing edge, and
- there are no bivalent vertices with one incoming and one outgoing edge.



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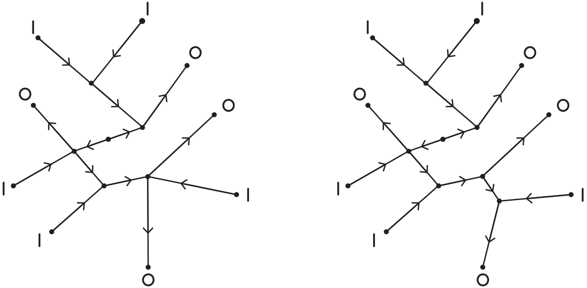
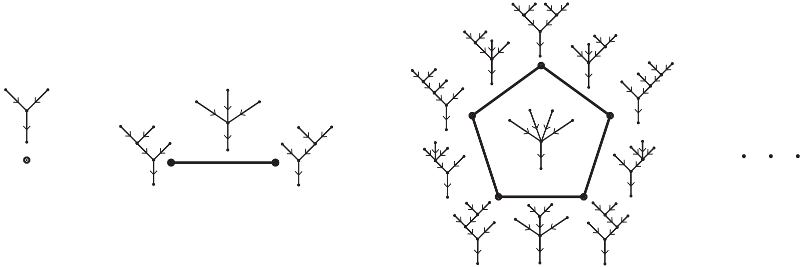


Definition

An *edge expansion* of a directed planar tree T is a directed planar tree from which T is obtained by contracting interior edges.

The space of directed planar trees



To build a space of edge expansions of a directed planar tree, we generalize Gelfand–Kapranov–Zelevinsky's secondary polytope construction of the associahedron.

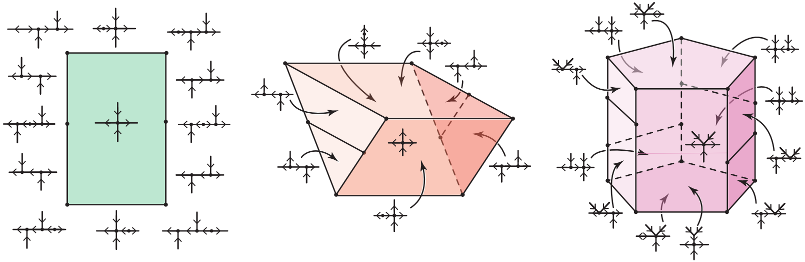
The space of directed planar trees

Theorem (P.–Tradler)

The space of expansions of a directed planar tree is a polyhedron. Faces correspond to edge expansions.

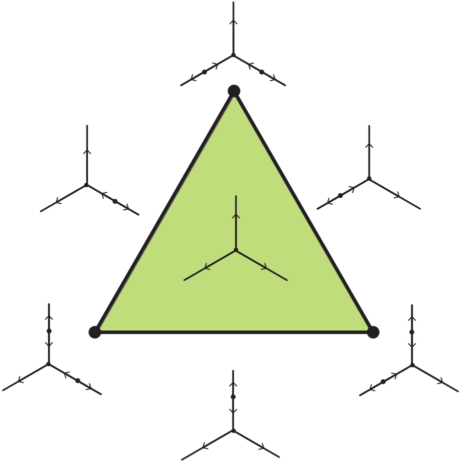
Definition 3′

The space of expansions of a directed planar tree which is a corolla is called an *assocoipahedron*.



The combinatorics of the assocoipahedra determine the structure of the $\mathcal{V}_\infty^{(d)}$ dioperad (Definition 0′).

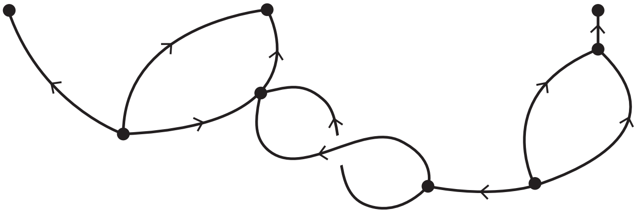
Assocoipahedron example



Directed graphs

Definition 2′

A *directed graph* is a directed fatgraph where each vertex with exactly one outgoing edge has at least two incoming edges.



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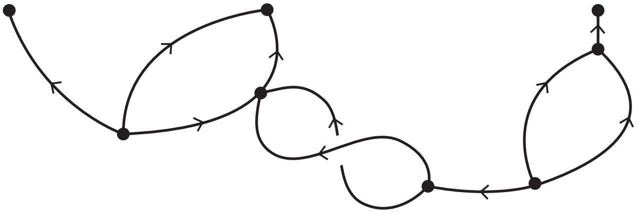


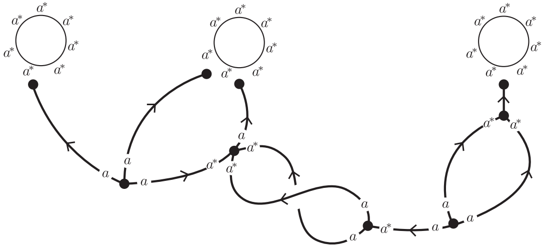
Proposition (Tradler–Zeinalian)

Directed graphs generate a chain complex $\mathcal{DG}_*$. The differential is the sum of edge expansions.

Directed graphs

Directed planar trees are building blocks of directed graphs. They keep track of elements of a $\mathcal{V}_\infty^{(d)}$-algebra. The chain complex of directed graphs $\mathcal{DG}_*$ is the space of algebraic string operations

$$CH^*(A)^{\otimes k} \to CH^*(A)^{\otimes \ell}$$



Corollary (P.–Tradler)

Assocoipahedra help us turn the chain complex $\mathcal{DG}_*$ into a cell complex $\mathcal{DG}$ whose complex of cellular chains is $\mathcal{DG}_*$.

Definition debt

Loop space definitions

✓ (1) short-branched trees
✓ (2) string diagrams $\mathcal{SD}$
✓ (3) space of short-branched trees
(4) slide complex

Hochschild definitions

✓ (0′) $\mathcal{V}_\infty^{(d)}$-algebra
✓ (1′) directed planar trees
✓ (2′) directed graphs $\mathcal{DG}_*$
✓ (3′) assocoipahedron
(4′) Stasheff quotient

Chapter 4: Main theorem

Loop space tree spaces versus Hochschild tree spaces

Main theorem motivation

Theorem (P.–Tradler, in progress)

A slide complex of short-branched trees is a decomposition of the Stasheff quotient of an assocoipahedron of type $(\bigcirc\bigcirc\cdots\bigcirc)$.

Type $(\bigcirc\bigcirc\cdots\bigcirc)$ means that edges adjacent to leaves in the directed planar trees are directed *outward* from the tree, toward the leaf.

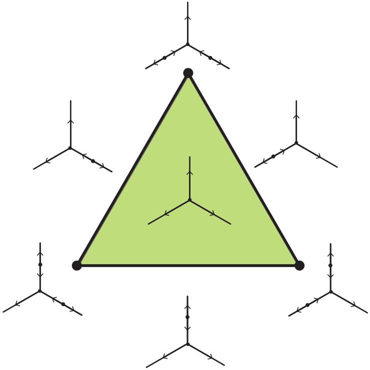
"Corollary" ("in progress")

There is a deformation retraction from the subcomplex $\mathcal{NDG}$ of $\mathcal{DG}$ consisting of of directed graphs with no directed cycles onto the space of string diagrams $\mathcal{SD}$.

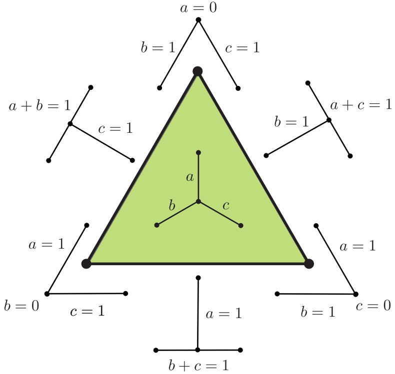
That is, the space of operations on $C_*(LM)$ is (homotopy equivalent to) a *subspace* of the space of operations on $CH^*(A)$.

Question for the future: Is there a larger space parametrizing operations on $C_*(LM)$?

Motivating example 1

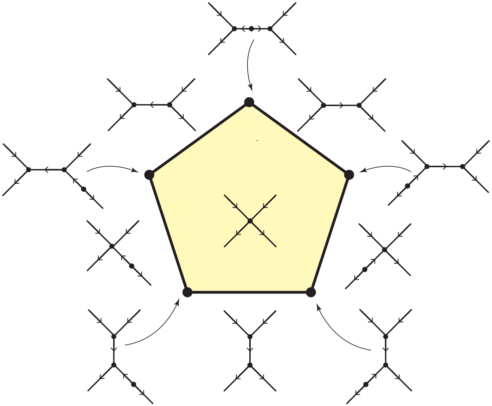


Assocoipahedron

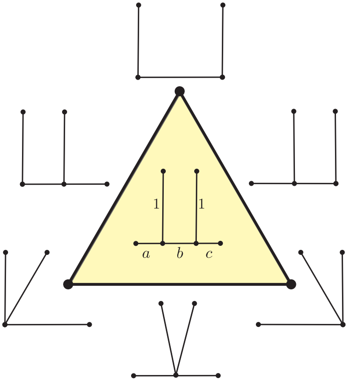


Space of short-branched structures

Motivating example 2



Assocoipahedron



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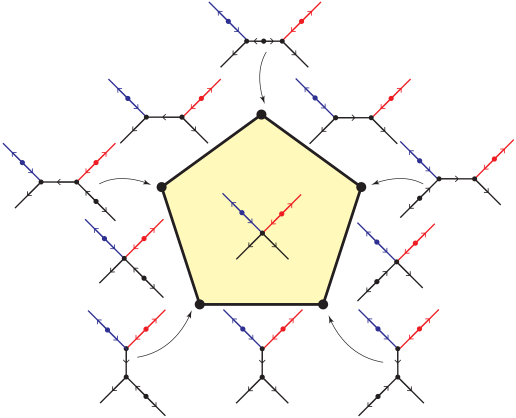
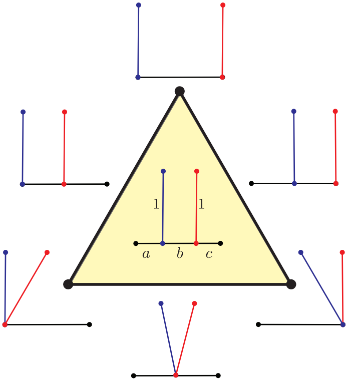
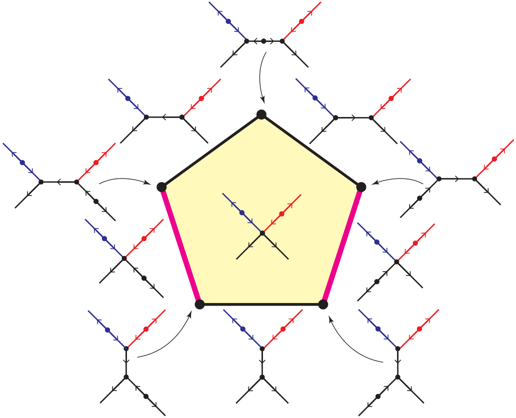
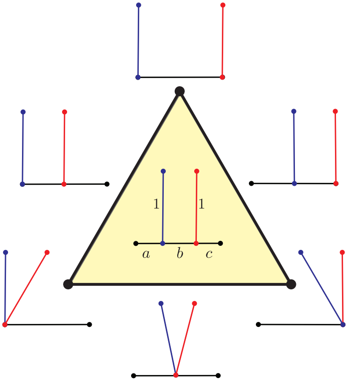




Motivating example 2





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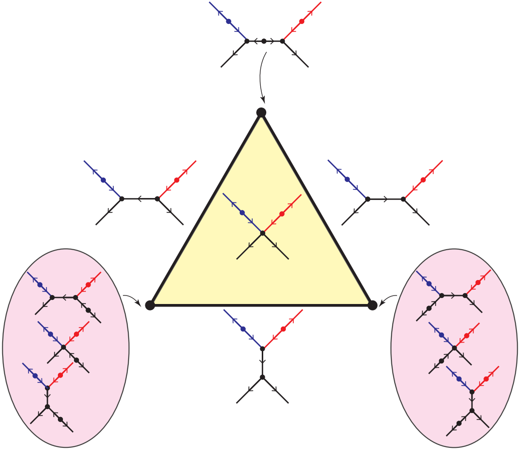
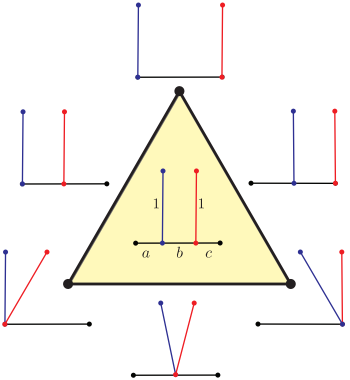
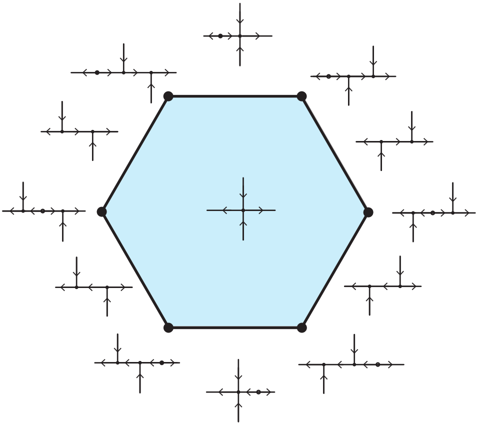
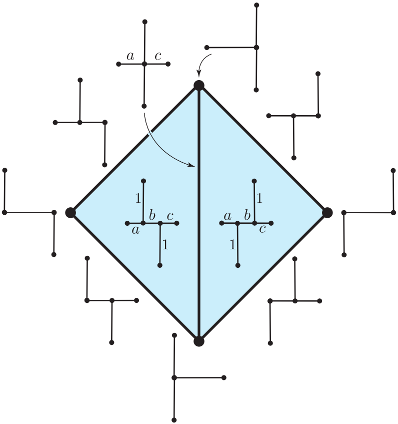




Motivating example 3

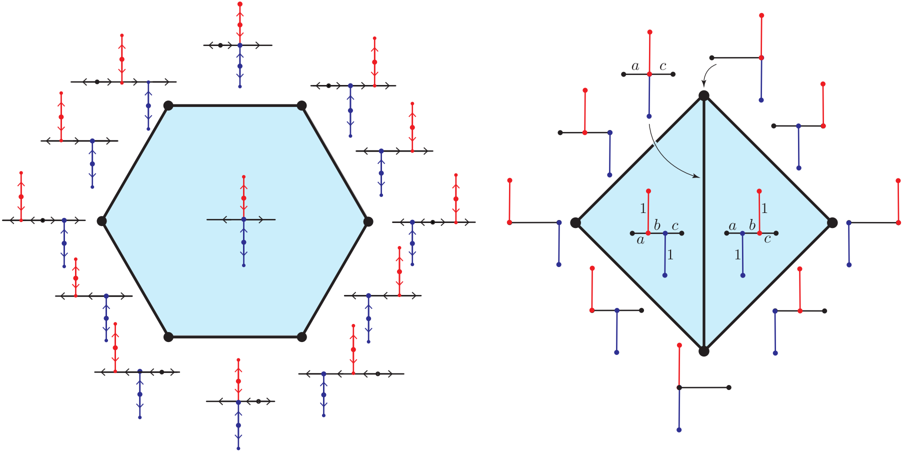


Assocoipahedron



Space of short-branched structures

Motivating example 3



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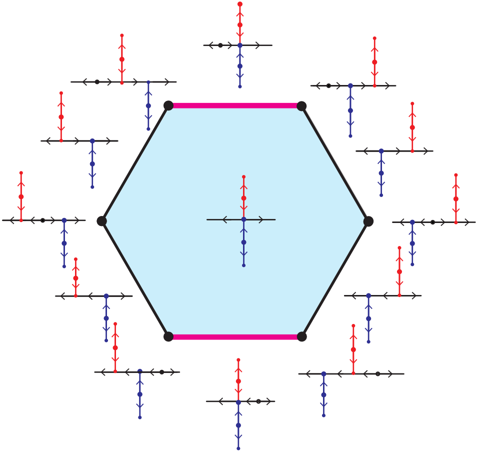

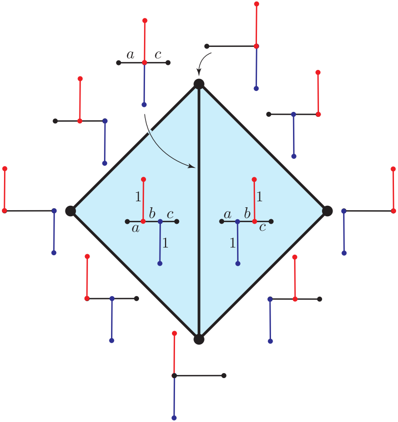

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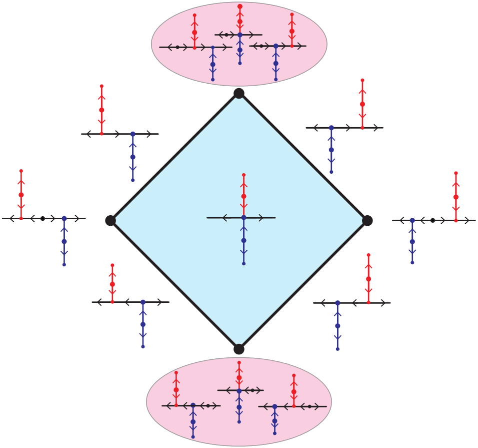

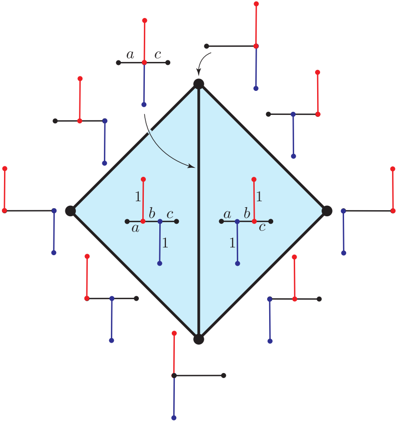

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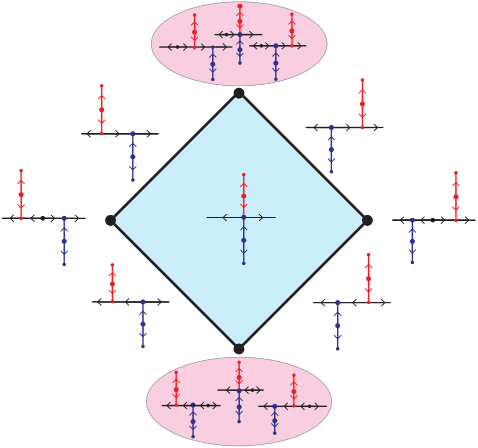
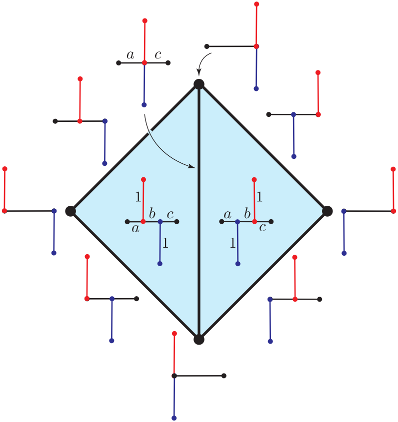


Stasheff quotient



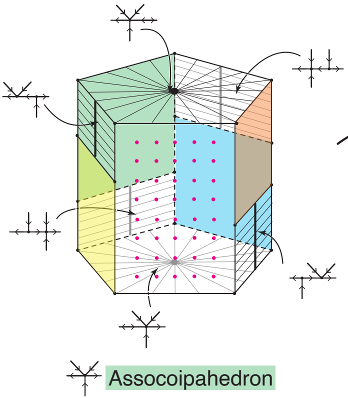
Slide complex

Final definitions

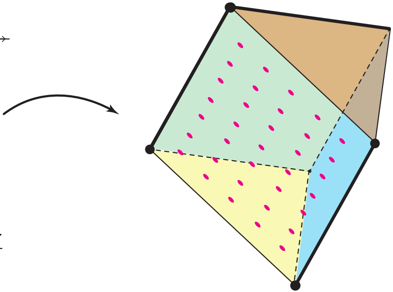
Each face of an assocoipahedron is a product of assocoipahedra; some factors may be associahedra.

Definition 4′

The *Stasheff quotient* of a assocoipahedron is the complex obtained by contracting associahedron factors of faces.



Assocoipahedron



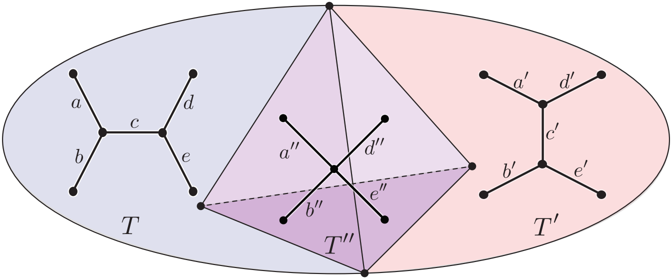
Stasheff quotient

Final definitions

Different spaces of short-branched trees may be identified along common faces. Top-dimensional cells are labeled by trivalent trees; those with a common codimension one face differ by a Whitehead move.

Definition 4

Given a fixed set of leaves L the *slide complex* is the space of all short-branched trees with L as their set of leaves.



Slide complex

Definition debt: paid off!

Loop space definitions

- ✓ (1) short-branched trees
- ✓ (2) string diagrams $\mathcal{SD}$
- ✓ (3) space of short-branched trees
- ✓ (4) slide complex

Hochschild definitions

- ✓ (0′) $\mathcal{V}_\infty^{(d)}$-algebra
- ✓ (1′) directed planar trees
- ✓ (2′) directed graphs $\mathcal{DG}_*$
- ✓ (3′) assocoipahedron
- ✓ (4′) Stasheff quotient

Theorem (P.–Tradler, in progress)

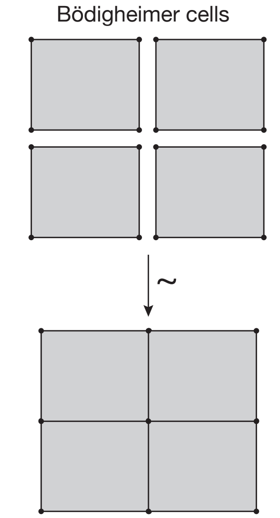
A slide complex of short-branched trees is a decomposition of the Stasheff quotient of an assocoipahedron of type $(\bigcirc \bigcirc \cdots \bigcirc)$.

Proof involves a *further* decomposition of both the Stasheff quotient and the slide complex 😬 🙄

Epilogue

Moduli space conjecture

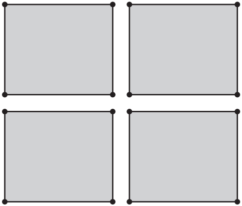
Moduli space of Riemann surfaces with boundary

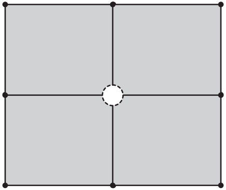


Harmonic compactification of moduli space of Riemann surfaces

Moduli space of Riemann surfaces with boundary

Bödigheimer cells

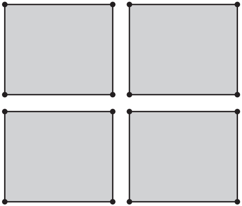


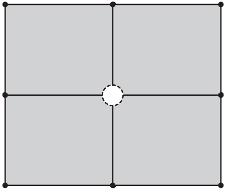


(Open)
moduli space of Riemann surfaces

Moduli space of Riemann surfaces with boundary

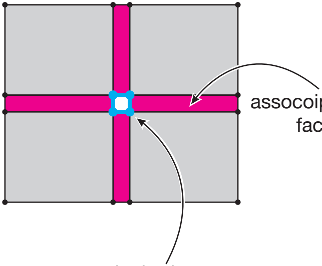
Bödigheimer cells





(Open)
moduli space of Riemann surfaces

$\mathcal{SD}$



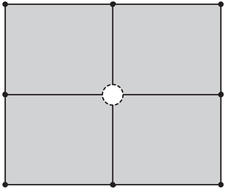
assocoipahedron
factors

assocoipahedron
with identification on boundary
(decomposed in $\mathcal{SD}$)

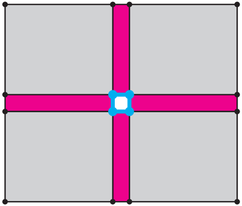
Moduli space of Riemann surfaces with boundary

Conjecture A

The space $\mathcal{SD}$ of string diagrams is homotopy equivalent to the moduli space of Riemann surfaces with boundary.



(Open)
moduli space of Riemann surfaces

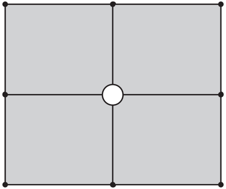


$\mathcal{SD}$

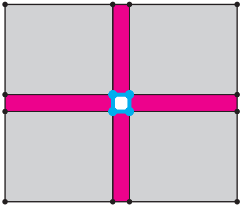
Moduli space of Riemann surfaces with boundary

Conjecture A'

The space $\mathcal{SD}$ of string diagrams is homeomorphic to the cut-off moduli space of Riemann surfaces with boundary.



(Cut-off)
moduli space of Riemann surfaces



$\mathcal{SD}$

Moduli space of Riemann surfaces with boundary

Recall:

“Corollary” (“in progress”)

There is a deformation retraction from the subcomplex $\mathcal{NDG}$ of $\mathcal{DG}$ consisting of of directed graphs with no directed cycles onto the space of string diagrams $\mathcal{SD}$.

Conjecture B

The space $\mathcal{DG}$ of string diagrams is homeomorphic (or at least homotopy equivalent) to the harmonic compactificataion moduli space of Riemann surfaces with boundary.

Then:

Corollary

The cellular chains on the harmonic compacticification of moduli space may be given the structure of a properad such that:

1. $C_*(LM)$ is an algebra over this properad, and
2. $CH^*(A)$ is an algebra over this properad.

THANK YOU